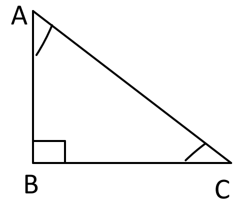
Trigonometric Functions/Equations

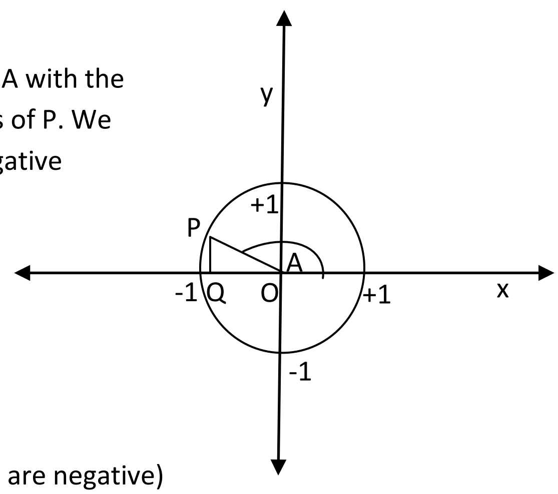
We studied trigonometric ratios in class IX/X. The idea of trigonometry originated to study triangles (trigos + metron). However, it was later generalized as a function and now has various applications. Let us see how the idea of sin(x) or cos(x) can be extended to values other than 0 to 90 °.

We learnt the ratio method of defining the trigonometric ratios:
sin(A) = BC/AC
cos (A) =AB/AC
tan (A)= BC/AB = sin(A)/cos(A)
cot (A)=AB/BC= 1/tan(A)
cosec(A)=AC/BC =1/sin(A)
sec(A)=AC/AB =1/cos(A)



But, can we define the value of sin (120°) or cos (210°)? Can we extend this idea to all possible angles from 0 to 360 °? The answer is yes, we use a different method, called the unit circle method. Consider a unit-circle (of radius 1 unit), as shown in the figure below.

Look at a point P on the unit circle, which subtends an angle A with the positive x-axis. Now, let us try to write down the coordinates of P. We can do that by dropping a line parallel to y-axis, onto the negative x-axis, we get a right angled triangle – OPQ.



$Let\ \angle OPQ = x$ (This is an acute angle)
sin(x) = PQ/AP = PQ (It is a unit circle, radius= 1 unit)
cos (x)= OQ/AP=OQ (AP= radius= 1 unit)
So PQ= sin(x) and OQ= cos(x)

So the coordinates of P are (sin(x), - cos(x)) (As x coordinates are negative)
Let us now try to extend the notion of sin(x) and cos(x) to all angles from 0° to 360°. Let us start with angle P, which lies in the second quadrant.
sin(A) is defined to be y-coordinate of P and cos(A) is defined to be x-coordinate of P and the other ratios follow from these definitions.
So, P = (cos(A), sin(A)) if A is the angle subtended by P, with positive x-axis and if it lies on unit-circle.

Question1: Does this definition agree with the ratios method definition which we discussed above? Does ratio method define trigonometric function values for angles other than those in the range 0° to 90°? So how do you check if these two methods (unit-circle and ratio-method) give the same answer? Can you start out with say 45°? How about 30° or 60°?

Question2: Okay, so now that you agree that both the methods agree with each, which is a more general way of defining trigonometry? What do you mean by general method? Which method works in more cases? Can you derive the ratio-method definition from the unit-circle method definition?

Question3: So the coordinates of any point P (on the unit circle, subtending an angle A) are (cos(A), sin(A)). Using this information, can you find the value of sin(180°) and cos(270°)? How about tan(180°) or cosec(270°) or may be sec(180°)? [**Hint:** Look at the coordinates if A is 180°, they are (cos(180°),sin(180°))]

Question4: If the value of sin(30°)=0.5, can you find the value of sin (150°)? If you draw 150° on the unit circle and locate the point P subtending this angle, what are its coordinates? Aren't they (cos(150°),sin(150°))? Can you locate coordinates of point P' (the cousin of P) who subtends an angle 30° with x-axis? Aren't they (cos(30°),sin(30°))? Can you find a relation between coordinates of the cousins P and P'? Is the x-coordinate of P' positive? Is the y-coordinate positive?

Question5: Can you now find the value of cos(150°)? How about tan(150°) and sec(150°)? If you know the value of sin(A) and cos(A), can you find the other trigonometric functions? If you know the value of sin(A), can you find the value of cos (A)? sin(A) and cos(A) are coordinates of a point on the unit circle, so how would they be related? Draw the right angled triangle and observe that the x-side is cos(A), while the y-side is sin(A), what about the hypotenuse? Is hypotenuse a radius of the circle, what is its length?

Question6: Locate a point P subtending an angle A=210° with the positive x-axis. Is sin(A) positive, what about cos(A) , how about tan (A)? [**Hint:** coordinates of P are (cos(A),sin(A))]. Using a method similar to that of Question5, can you find the values of sin(210°),cos(210°), tan(210°) and sec(210°)?

Question7: If you have to solve sin (x)= 0.5, what are the answers you can think of? How many answers does it have? Can it have an answer in 1^{st} quadrant? What about 2^{nd} quadrant? What about the 3^{rd} quadrant? What about 4^{th} quadrant? What is the sign of sin(x) in first quadrant (positive/negative)? Can it have more than one answer in first quadrant? Do you have more than one point P, on the unit circle, whose y-coordinate is equal to 0.5? So how many solutions can you find?

Question8: We have defined values of trigonometric functions of angles between 0° to 360°. How about other angles? Is 390° equal to 30°? Why is 360° considered to be one full rotation? So what is the value of sin (390°)? What is the value of sin (570°)?

Question9: What about values of negative angles? Why is anti-clockwise considered positive? Can you locate a point P which subtends an angle -30°, with the positive x-axis? What is the value of sin (-30°)? A function f is called even function if f(-x)=f(x) and odd function if f(x)=-f(-x)? Which trigonometric functions are even functions, which of them are odd functions?

Question10: You are required to solve sin(x)=0.5 (x is any real number). If x= 30° is one answer, would 390° be an answer an aswell? What are the various answers you can think of? Does 750 work as an answer? How about a (multiple of 360°) + 30°? [Hint: One- go back to Question7 to review the two answers which you found.]

Radian System of measuring angles

We used the degree system and not the radian system, just so that you're not confused. Now let us look at the radian system. The motivation for radian system is quite simple. Look at an angle A subtended by a point P (on the unit circle), with the positive x-axis. [If you can't visualize this, draw it out on a piece of paper. In the long run, with practice you should be able to visualize these problems, without pen and paper. This exercise improves your speed as well as conceptual clarity]

Look at the length of the arc XP (where X=(1,0)), it's length is equal to (A/360°)*2 *π)units as the radius is equal to 1unit. This is because the length of the arc is proportional to the angle subtended by the arc at the center of the circle. Here is A is measured in ° and we have too much work to do. Don't you think this formula is scary? Can we measure the angle in terms of the length of the arc? So, case 360 ° = 2π in new system, which is called as radian-system! 360° = 2π, 180° = π, 60° = $\frac{\pi}{3}$ and 45° = $\frac{\pi}{4}$. What about 30°, can you convert it to radian system?

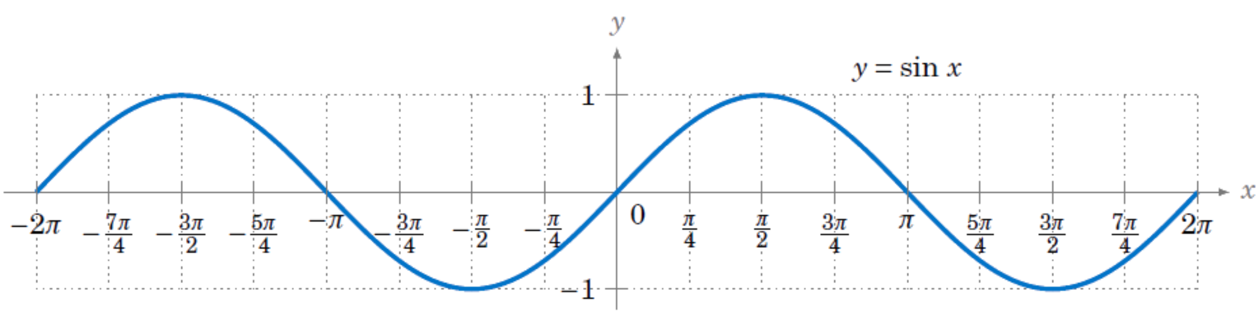
Question1: Can you now rewrite your answers (of the previous ten questions) in radian system? [You needn't solve them all over again, you can just change the units of the answers into radian system]

Question2: If you are confused about the radian system, but can solve the question in degree system, what can you do? Can you find all the answers of sin(x)=1?[**Hint:** Can you try the problem reduction technique? If a badminton player isn't good at backhand, how doe he/she play? Does he/she move and play forehard?]

Question3: Why is $\frac{\pi}{2}$ equal to quarter circle? Why is 90° equal to quarter circle? Can this be used to visualize 90° as $\frac{\pi}{2}$radians? Why is $\frac{\pi}{3}$ equal to one third circle? Why is 120° equal to one third circle? Can you use this method to convert 60° into radian system? How about 30°?

Question4: Is sin(7 $\pi/6$) positive? How about cos (6 $\pi/7$)? What about tan (7 $\pi/8$)?In which quadrant do these angles lie? Can you find the sign of sin(x) and cos(x) once you know the quadrant in which x lies? [Can you use the coordinates to find this out] Once you know the sign of sin(x) and cos(x), can you then find the sign of the other four trigonometric functions? [**Note:** Many students get confused with these kind of problems which ask you to find the sign of a trigonometric function-they complicate their life by ignoring the basics and beating around the bush].

Question5: Look at the graph of sin(x). Can you use the graph to find all the answers of sin(x)=1/$\sqrt{2}$? If you know all the answers between 0 to 2π, can you then find all answers from -∞ $to + \infty$? [Problem Reduction technique] How many answers does each trigonometric function have for f(x)=0.5 [f can be sin,cos,tan,sec,cosec or cot]? Why only two solutions?



Commentary

Elon Musk uses the Feynman Technique(being able to explain concepts to a five year old, by replacing harder words with easier words) to solve problems- he calls it the physics way of problem solving. For instance, if someone would ask Feynman " why do shoe-soles wear out", he would tell you that the rough edges of your shoe hit the rough edges of the ground and fall off slowly. He doesn't hide behind labels like "friction". Feynman says that there is a difference between knowing the name of something and understanding something.

Most students find trigonometric functions intimidating. However, we guys had derived them from the scratch- as though no mathematician discovered it and we were the first ones![Sciensation asks- "Did God enter Newton's dreams and tell him "beta equation note karlena, physics aisa chatla hain"?] Elon Musk calls it problem solving from the first principles. So shall we learn Trigonometry using the method used by one of the most revolutionary entrepreneurs ever- the Founder of SpaceX/Tesla Motors/HyperLoop?