Sciensation Society,
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Ramanujan Mathematics Dialogue

General Instructions

- Students can send in their research write-ups in teams of three
- There are two parts- Analysis and Synthesis
- All the questions in first part (Analysis) are compulsory, for required subject background.
- Students can choose any two themes and send in their research reports.
- The team which would have submitted the best research shall present during the event.
- One team per theme would be selected. You may choose the theme strategically.
- Describe the background work which was done to complete the research report.
- Interesting wrong answers are valued more than correct answers, at Sciensation!
- If you disagree with any implicit assumptions in the question, please state your point of view explicitly. You'd receive extra marks.
- Sciensation is very generous when it comes to awarding points, creative answers receive extra-marks!

Selection Criterion

- Examples- Can the student provide intuitive examples to justify the logic?
- Eloquence- Were the arguments explained carefully?
- Rigor- The arguments/assertions need to be justified.
- Robustness- Can the argument break down easily? How general is it?
- Elegance- Were the arguments beautifully constructed?

Scoreboard

	Examples	Eloquence	Rigor	Robustness	Elegance
Analysis: P1	/5	/5	/5	/5	/5
Analysis: P2	/5	/5	/5	/5	/5
Analysis: P3	/5	/5	/5	/5	/5

	Examples	Eloquence	Rigor	Robustness	Elegance
First Theme	/5	/5	/5	/5	/5
Second Theme	/5	/5	/5	/5	/5

(Will be considered under both the themes)

Total (Analysis + Synthesis):

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Analysis Part One: Abstraction

Mathematics deals with abstraction- when meaning moves away from a physical form. So how do we understand these abstract objects?

Q1: What objects do we use to count number of chocolates? What if we break these chocolates into pieces? Why do we invent fractions? Why do we talk of parts of a whole? Can fractions be added? Why do they inherit addition, multiplication, subtraction, division from numbers- why can we perform all these operations, just like how we would, with natural numbers? Do we use natural numbers to understand fractions?

Q2: Ramanujan has thought of a number and he won't tell you that number, he will only give you clues. He tells you that when his number is added to 20, he gets the same answer when his number is multiplied with itself? Why do we call his number x? Why do we write x + 20 = x * x? If this is scary, can we write it as Ramanujan + 20 = Ramanujan * Ramanujan? Why would we use alphabets to label variables?

Q3: Ramanujan now starts changing the number in his mind. His friend and mentor Hardy also starts changing his number is his mind. Ramanujan tells you that no matter how many times he changes his number, we have the result $(\text{Ramanujan} + \text{Hardy})^2 = (\text{Ramanujan})^2 + (\text{Hardy})^2 + 2*$ Ramanujan * Hardy. Do we look at variables as unknown or do we look at them as containers of values? Do we have to know their values, to do Mathematics?

Q4: Arybhatta number of students went to a shop and everybody purchased the same items- Bhaskara kg of apples for Rs Hemachandra per kg and Madhava litres of milk for Rs Varahamira per liter. If Rs Ramanujan was the initial amount of money with the group, how much money is left with them? Replace these variables(with names of legendary Mathematicians) with values and discover the logic. Now replace the numbers with variables. Why do operations of numbers appear easier than those with variables?

Q5: If we can work with numbers alone(i.e. reduce questions around variables and fractions to those on natural numbers), as seen in the above four questions why do we have to learn these higher order concepts? Why do we build up ideas on top of these ideas? These ideas may be simple for the one who created them as he/she can see why they were created and how it reduces effort, but how do we learn abstract ideas?

A2: Reading Proofs in Mathematics

Q1: Algebraic symbols: We generally use symbols to reduce the length of description. It can usually be a little intimidating at the beginning, but once you get used to it, they're bread and butter stuff. It is just like a new language- you'd initially translate to English, but once you get used to it, it's simple. For instance $\log_2 1024$ basically asks, to what power should 2 be raised so that it is equal to 1024 or how times should 2 be multiplied so that it is equal to 1024. Can you find $\log_3 729$? Can you write down your own logarithmic equation? Is it hard?

Q2: Equation: What does an equation mean? When do you say that two things are equal? Do I need an equivalence criterion for answering that? Would Rahul Gandhi be equal to Modi if the equivalence criterion is that "all Indians are equal" and a non-Indian is never equal to an Indian? Now look at the example take the equation a = b mod p, according to this special kind of equations called Modular Arithmetic, the equation is valid if a and b leave the same remainder when divided by p. For instance 9 = 5 mod 4 = 13 mod 4 = 1 mod 4 and so on.

Q3: Deduction Keywords- If, Only If, If and Only If, Necessary, Sufficient, Necessary+Sufficient
Statement1: You will pass the exam if you understand the complete syllabus.
Statement2: You will pass the exam only if you score more than 40% marks.
Statement3: You will pass the exam if and only if you are conceptually clear.
Label the above statements as necessary, sufficient or both, provide the required explanation.

Q4: Deductions: All Sciensation students are smart and All Smart students can solve problems implies All Sciensation students can solve problems. What does "implies" mean? Consider the statement A implies B & B implies A, is it equivalent to A is true iff B is true? We know that B is true implies A is true and vice-versa but how do we know that B is false implies A is false? Does it also imply that B is true iff A is true? Does it also imply that A is false iff B is false?

Q5: Feynman Technique: How do you convert the arguments to simple english, with ideas and examples which can be followed by a five year old child? Use it to understand the proof given below. If a = r mod p and b = s mod p then a*b = r*s mod p.

a leaves a remainder r when divided by p or a coins when grouped into bags of size p leave out r coins behind. Likewise b leaves a remainder s when divided by p. We need to find the remainder when the product of a and b is divided by p.

a = mp + r and b = wp + s (by using division algorithm, dividend = divisor * quotient + remainder)

So a * b = (mp + r) * (wp + s) = mwpp + mps + wpr + rs = p * SOMETHING + rs,
So, ab - rs = p * SOMETHING and if the difference of two numbers is a multiple of p, both numbers have to leave the same remainder when divided by p (can prove it by contradiction).

Analysis Part Three: Proof Writing

Mathematics Requires a rigorous proof for its arguments. E.T.Bell had said that obvious is the most dangerous word in Mathematics. Bertrand Russell and Whitehead wrote a proof for 1+1 =2 which runs over 200 pages. Let us understand how beautiful proofs are written in Mathematics.

Q1: Introduction of variables and constants: Do we understand the difference between a variable and a constant? What values change in the system? What variables/constants are important and worthy of being observed/studied? For instance, in an Algorithmic context, we talk of the size of the input n and the number of steps needed to solve n O(n). Or in a number theoretic context, we define a number n and its prime factorized form $n = p^a q^b r^c$

Q2: Constructing rigorous arguments/definitions: How do we present the argument/definition in a easy to understand and precise manner? For instance, somebody says p is a prime number, a rigorous way of stating that is a divides p implies a =1 or p for any natural number a. A rigorous way of stating that the set S is a subset of set M: x in S implies x in M for all x.

Q3: Understanding the flow of arguments: Can you think of the arguments, without the variables or equations, just to understand how the structure of the proof? For example consider the proof for the fundamental theorem of Arithmetic- prime factorization of a natural number is unique.
A1: Every prime factorized form of a number n contains the same primes.
A2: Every prime factorized form of a prime number contains the same exponent for every prime.
A3: Proof of A1 and A2 implies that prime factorized form is unique.

Q4: Understanding Rigor: Does an argument require a proof? If you can produce a few examples does it suffice to prove an argument? Can you play the devil's advocate? Is the deduction supported by another theorem or another proved argument?
For instance, in Q3, although it may appear obvious, we need to prove that every list of factors contains the same primes. We can prove it by contradiction, let us assume that one such prime doesn't exist in one list and then contradict the factor that the product of all the primes in the other list is equal to n.

Q5: Completing a proof: Start by thinking of the arguments and then try to organize your proof by assuming the required variables and constructing the required arguments. For instance, look at this proof for $S_n = 2S_{n/2} + n \Rightarrow S_n = n * (\log_2 n)$.

2Sn/2=4Sn/4 + 2*(n/2)

4Sn/4=8Sn/8 + 4*(n/8)

....

n/2S2 = nS1 + (n/2)*2

All terms except Sn and the n terms get cancelled, and number of n's = log n, hence proved

Question for Discussion: Take any one mathematical statement and write a proof

Synthesis Theme1: Number Theory

Q1: Fermat's Little Theorem: Consider a natural number a and a prime number p such that p ,a are relatively prime, then a^{p-1} leaves a remainder 1 when divided by p. How would you understand this statement using Feynman Technique? How would you understand a multiplied p-1 times? Can you take some examples of a and p and visualize? Can you draw the sizes of the bags and visualize how the height of the bags increases?

Q2: What do you mean by relatively prime? If there are no common factors for a and p, can you conclude that there are no common factors for t*a and p, where t is some number from 1 to p-1? Let us use toy example technique here, let p=5 and a=2, so t ranges from 1 to (p-1)5-1=4, so we consider 1*2, 2*2, 3*2,4*2, and our claim is that none of these numbers are divisible by 5. Can we observe that 2 is not divisible by 5 and 1,2,3,4 are not divisible by 5? Let us look at one more example, let p=7 and a=4, we get the numbers 1*4, 2*4, 3*4,4*4,5*5,6*4, are any of these numbers divisible by 7? Can you write the numbers for a=6 and p=11 and check the claim?

Q3: 1,2,3..p-1 are possible remainders when 1*a, 2*a,3*a...(p-1)*a are divided by p, what if we show that each remainder is unique? Can we conclude that each number shows up at least once? Can we write all the numbers 1*a, 2*a, 3*a... (p-1)a and say that each possible remainder 1,2,3..p-1 has to go into one particular block? Let us take an example, a=2, p=5, 1*2=2, 2*2=4, 3*2=6, 4*2=8, look at the remainders when these numbers are divided by 5: 2,4,1,3 - we have got back all numbers 1,2,3,4 in a jumbled form. Can we see this pattern with a=6 and p=11?

Q4: How do we prove that the remainders are unique? By contradiction? Assume that two remainders, say m*a and n*a yield the same remainder, i.e. m*a=p*w +r and n*a=p*u +r, so m*a - n*a = (w-p)p then (m-n)*a is divisible by p. Let us take a toy example, if 2*6 and 9*6 leave the same remainder when divided by 11, if we subtract the smaller number from the bigger one, the loose change should also get knocked off, so we're left with bags of 11, so 7*6 is divisible by 11, which is not possible as seen in Q2, so the remainders left by 2*6 and 9*6 are different.

Q5: We have the numbers 1*a, 2*a, 3*a, 4*a... (p-1)a, they have all the possible remainders 1,2,3... p-1 amongst them. If we multiply these numbers, we get 1*2*3...(p-1)* a*a*....a = 1*2*3...(p-1)*a(p-1) on the LHS and we get 1*2*3*4...(p-1) on the RHS as the remainder. If we cancel 1*2*3...(p-1), then we get a^{p-1} leaves a remainder 1 when divided by p.

Question for discussion: Critique Euler's observation: Fermat didn't really require p to be prime, he needed 1,2,3...p-1 to be relatively prime with p. So Euler replaced 1,2,3..p-1 with the numbers which are relatively prime with n. For instance, if n=10, he took 1,3,7,9 are relatively prime with 10. Same arguments repeat- their remainders are unique, so they have to 1,3,5,9 in a jumbled order, so we get a^4 leaves a remainder 1 when divided by 10 whenever a is relatively prime to 10. Does this answer periodicity of powers-7: 7,(4)9,(34)3,(..)1,, 8: 8,4,2,6 etc.

Synthesis Theme2: Artificial Intelligence

Q1: If you build a tree of all possibilities this way and if you'd be given the answers dynamically, can you use the tree to design your algorithm? Construct a tree for the instructions given below?

Stage1: Do you see red(move left) or yellow(move right) or move straight otherwise
Stage2: If it was red, you would taken the left turn, now, if you see blue take left, if green take right, move straight otherwise.
Stage3: If it was yellow, you would have taken right, if you see orange now take left, if you see purple take right, move straight otherwise.
If you have such a tree, can you design an algorithm?

Q2: Can you construct a goal tree by breaking a goal into smaller parts and those parts into parts and so on. For instance, if we have a goal of making a Cake, we have three subgoals of making the cream, baking the flour for the base and the toppings for decoration. The goal of topping can be subdivided into placing cherries, cut fruits and wafers. The goal of cut fruits can further be subdivided into purchasing fruits, cleaning fruits, cutting fruits and removing the seeds. Working upwards helps us in achieving the goals, by focusing on one thread at once.

Q3: Can you construct a goal tree to guide a Chinese restaurant's automated telecaller in answering calls? How would you understand the possibilities? Can you go through a few dry runs to understand possible questions? What if the question isn't understood by the algorithm? Can you get a human being to take that call? How would you ensure that the client doesn't know that it was algorithm which took the phone call and that it transfered the call to a human?

Q4: Can you design your algorithm in such a way that it learns from previous information? Once a call is returned to a human, how can your algorithm make sure that future calls are handled by the computer itself? If most of your calls are regarding a certain inquiry, say "do you have an offer on pizzas today?", would you be able to improve your algorithm so that the time requirement is reduced. What do you understand by automation? How do you identify repetitions so that you could automate them? Can you list various kinds of repetitions?

Q5: What is Intelligence? What is Artificial Intelligence? Prof Patrick Winston of MIT, in his opencourse on AI says that Intelligence ceases to appear intelligent after you figure out how it works. So does a chatbot appear intelligent now? How does one make an AI model for something? Can a computer really think? Can you list tasks which can be automated?

Question for discussion: Pick up a task which can be automated and draw a decision tree to automate it. Explain the Mathematics used.

Synthesis Theme3: Economics

Q1: We see choices and outcomes in different situations of life. They can be modeled as games. For instance, let us at look at the most famous game in game theory- the Prisoner's Dilemma. There are two prisoner in two different cells and they have two choices- he either confess or he doesn't and the consequences depend on both the players' choices. What should the prisoner do? Should he confess or should he not?

	B confesses	**B doesn't confess**
A confesses	(A: Prison: 3yrs, B: Prison: 3yrs)	(A: Free, B: Prison: 5yrs)
A doesn't confess	(A: Prison: 5yrs, B: Free)	(Prison: 1year, Prison: 1year)

Q2: Consider another game. with Ramu and Shyamu taking up a group assignment. If one does the homework, he takes up all the effort and the other benefits. If both give up, they both lose marks. If both work, they receive better marks but the effort is higher. Is this game comparable to Prisoner's Dilemma? Consider another game- a business sells a product to a client. If the product is faulty, the client loses and the business profits. If the client doesn't pay, the business loses and the client gains. If the product is good and if the client pays, both benefit. If the product is bad, client doesn't pay, both lose. Is this game comparable to Prisoner's Dilemma. Can you think of games similar to Prisoner's Dilemma? Do these games come with features? Understanding real life games makes sense, but why analyze these Mathematical games?

Q3: Can we design games to induce the desired behavior? If you work for the government and you require companies to place tenders for a coal mine, what would your objective be? How do you ensure that you sell the rights for the highest possible price? One common example is an auction, but what if the second highest bidder has a budget of Rs100 and highest bidder has a budget of Rs200, you would get a price of Rs101 although somebody is willing to pay Rs200, would that be okay? How would you design the game so that the highest bidders bids high?

Q4: Would tender (also called sealed bids) procedure yield the 200 bid? What if we change the game and say that the highest bidder gets it at the price of the second highest bidder(Vickrey acution)? Would people increase their bids? Wouldn't every player think that the other bidders would increase their bids? Wouldn't highest bidder question himself- will the second highest bidder bid 100 or more? Will the whole system think this way? Are the bidders/contractors smart people? So should the IAS officer designing the tender process be smarter?

Q5: Can a teacher use mechanism design to make students better? Why does sciensation give points for both questions and answers? Can demonetization be understood as reverse game theory? **Question for discussion:** Can you use this idea of mechanism design to design a game which induces the required behavior? Explain the Mathematics used.

Synthesis Theme4: Dynamic Programming

Q1: Should we start working after planning or should we plan after starting to work? In a Telugu movie "Julayi", the director Trivikram Srinivas gets his star cast Allu Arjun to say "Generally smart guys plan and then start working, but then street smart guys plan while working". Can Dynamic planning be a good strategy? Can our algorithm get formed while it is working?

Q2: Prime Factorization: You need to find the largest prime less than n=10,000,000,000. Would you test every number less than n? Every prime less than n? Would all primes <= square root of n suffice? If all primes <= square root of n don't divide n, can the product of two primes > square root of n be equal to n? Can product of two numbers greater than 10 be equal to 10? Sounds cool, but wouldn't you need all primes <= square root of n, how? Can we dynamically list that? Initial prime list: 2,3. Try 5, it is not divisible by 2, so add 5 to the list, try 7, it is not divisible by 2 and continue this way to build the list until you reach n. Why use dynamic programming here?

Q3: Example2: Look at the towers of Hanoi or towers of Brahma puzzle. We have three rods and a number of disks of different sizes. The disks can slide onto any rod. Initially, all the disks are arranged in an ascending order with the smallest disk at the top. The goal is to move all the rings from one stack to another stack. **Rules:** You can move only one disk at a time. You can only remove a disk which is on top of one of the stacks and you can place on top of another stack, any other position is not valid. No disk can be placed on a smaller disk. Can you first two disks and then use that solution to solve first three and so on? Can you prove that $S_n = S_{n-1} + S_{n-1} +1$ i.e. $S_n = 2S_{n-1} +1$? Can we prove $S_n = 2^n -1$? Why do we use Dynamic Programming here?

Q4: You are given a list of numbers and you're asked to sort them. Say 45, 87,34,23,89,11, 23,97,54,22,82,76,41,67,37, 89. Can you use dynamic programming to sort them? How can you break them into smaller problems and then reassemble a solution from the solutions of the smaller problems? A critical question to ask here is how do you assemble two solutions- do you assemble 23,89 and 22,54? Does it take a maximum of three steps? If each list has 100 numbers, how do you assemble their solutions? Does it take a maximum of 100 steps? If our initial problem had 2100 numbers, can we break into 299 lists of size and then sort them? So $S_n = S_{n/2} + S_{n/2} + n$? So $S_{16} = 2S_8 + 16$, $S_8 = 2S_4 + 8$, $S_4 = 2S_2 + 4$, $S_2 = 2S_1 + 2$, So S_{16} = 8*4=32 or $S_n = n * \log n$. How can you use Feynman Technique to understand this? n = 1024?

Q5: Dynamic Programming is also called memorization: breaking problem into smaller problems, solving, remembering their solutions and assembling them into a solution of the bigger problem. Do we solve our school math problems this way? Do we use solutions of lower grade problems? Is dynamic programming a way of thinking which we are already aware of?

Question for Discussion: Pick a problem and suggest a dynamic programming solution for it.